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Solvable Lie algebras with Abelian nilradicals†

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Abstract. A procedure is presented for classifying solvable Lie algebras with Abelian nilradicals. Theorems on the structure of such algebras are proven and their centres are constructed. Many examples are analysed.

Résumé. Une méthode est présentée pour classifier les algèbres de Lie résolubles avec nilradicaux Abéliens en classes d'isomorphie. La structure de ces algèbres est analysée et leurs centres sont construits. Plusieurs exemples sont analysés.

1. Introduction

The purpose of this article is to present some general results on the structure of finite-dimensional solvable Lie algebras with Abelian nilradical NR, over the field of complex numbers $\mathbb{C}$. The motivation for such a study is multifold. Indeed, the Levi theorem [1, 2] tells us that any finite-dimensional Lie algebra L can be decomposed in a unique manner into a semi-direct sum of a semi-simple Lie algebra S and a solvable ideal R, its radical:

$$L = S \triangleright R \qquad [S, S] = S \qquad [S, R] \subseteq R \qquad [R, R] \subset R\,. \tag{1.1}$$

The classification of solvable Lie algebras is thus an essential step in the classification of all finite-dimensional Lie algebras over fields of characteristic zero. The classification of all semi-simple Lie algebras over the fields of complex or real numbers, due to Cartan and Gantmakher, respectively, is of course classical and can be found in any book on the subject [2, 3].

From the point of view of physical (and other) applications we note that solvable Lie algebras often occur as Lie algebras of symmetry groups of differential equations. Algorithms exist for determining these symmetry algebras [4, 5]. Sub-algebras of the symmetry algebra can then be used to perform symmetry reduction, i.e. to construct group-invariant solutions. An important step in this procedure is to identify the symmetry algebra (and its sub-algebras) as abstract Lie algebras. Again, algorithms exist for realizing such an identification in broad terms (like decomposing L into a direct sum, or obtaining its Levi decomposition, etc) [6]. A detailed identification presumes the existence of a classification of Lie algebras into isomorphy classes.

For solvable Lie algebras such classifications exist only for low dimensions [7–10]. A recent article provided a classification of all solvable Lie algebras with Heisenberg algebras

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as nilradicals (maximal nilpotent ideals) [11]. Many important general results on solvable Lie algebras are due to Maltsev [12].

In this article we concentrate on the case when the solvable Lie algebra L has an Abelian nilradical. This case is of particular importance in applications to partial differential equations (and hence to physics). Indeed, the presence of Abelian sub-algebras of the symmetry algebra is often due to the fact that the equations have constant coefficients, or can be transformed into equations with constant coefficients.

In section 2 we present some general results on the structure of the Lie algebras L. In section 3 we discuss the decomposability or indecomposability of the obtained algebras. Section 4 is devoted to the centre $C(L)$ of L and in particular we obtain saturated bounds on the dimension of the centre. In section 5 we present some examples.

2. General structure of the Lie algebra

2.1. General concepts

A Lie algebra L is *solvable* if its derived series DS terminates, i.e.

$$DS = \{L_0 \equiv L, L_1 = [L, L], \ldots, L_k = [L_{k-1}, L_{k-1}] = 0\} \tag{2.1}$$

for some $k \geqslant 0$.

A Lie algebra is *nilpotent* if its *central series* CS terminates

$$CS = \{L_0 = L, L^{(1)} = [L, L], L^{(2)} = [L, L^{(1)}], \ldots, L^{(k)} = [L, L^{(k-1)}] = 0\} \tag{2.2}$$

for some $k \geqslant 0$.

Every solvable Lie algebra L has a uniquely defined (upto equivalence) maximal nilpotent ideal, called the *nilradical* $NR(L)$. The dimension of the nilradical satisfies [8]

$$r = \dim NR(L) \geqslant \frac{n}{2} \qquad n = \dim L\,. \tag{2.3}$$

Any solvable Lie algebra L can be written as the algebraic sum of the nilradical $NR(L)$ and a complementary linear space F

$$L = F \dot{+} NR(L)\,. \tag{2.4}$$

The derived algebra $DL = [L, L]$ of a solvable Lie algebra is contained [2] in the nilradical $DL \subseteq NR(L)$.

An *element* n of L is *nilpotent* if it satisfies

$$[\cdots[[x, n], n]\cdots n] = 0 \qquad \forall x \in L$$

when the commutator is taken sufficiently many times.

A set of elements $\{x_1, \ldots, x_k\}$ of L is called *linearly nilindependent* if no non-trivial linear combination of them is nilpotent.

Use will be made of the adjoint representation $\operatorname{ad} L$ of L given by

$$\operatorname{ad} l \cdot x = y = [l, x] \qquad x, y, l \in L$$

and of the restriction of $\operatorname{ad} L$ to the nilradical of L: $\operatorname{ad}_{NR} l$. This restriction $\operatorname{ad}_{NR} l$ is realized by matrices $A \in K^{r\times r}$ where K is the ground field assumed to be of characteristic zero. If l is a nilpotent element of L, it will be represented by a nilpotent matrix in any finite-dimensional representation. In particular $\operatorname{ad}_{NR} l$ will be nilpotent.

A set of *matrices* in $K^{r\times r}$ will be called *linearly nilindependent* if no non-trivial combination of them is nilpotent.

From now on we shall assume that $NR(L)$ is Abelian and that the algebra L is *indecomposable*, i.e. cannot be decomposed into a direct sum of two (or more) sub-algebras. The algebra L is finite-dimensional: $\dim L = n$. We shall choose a basis for the nilradical and for the space F, putting

$$NR(L) = \{n_1, \ldots, n_r\} \qquad F = \{x_1, x_2, \ldots, x_f\} \qquad r + f = n \qquad \frac{n}{2} \leqslant r \leqslant n-1\,. \tag{2.5}$$

We thus also assume that L is not nilpotent ($NR(L) \neq L$).

2.2. *Basic Structural Theorems*

Theorem 1. Let L be a finite-dimensional solvable non-nilpotent Lie algebra over the field K and let its nilradical $NR(L)$ of dimension r be Abelian. Then we can choose a basis (2.5) of L such that the commutation relations are

$$\begin{pmatrix} [n_1, x_\alpha] \\ \vdots \\ [n_r, x_\alpha] \end{pmatrix} = A^\alpha \begin{pmatrix} n_1 \\ \vdots \\ n_r \end{pmatrix} \qquad 1 \leqslant \alpha \leqslant f \leqslant \frac{n}{2} \qquad A^\alpha = -(\underset{NR}{\operatorname{ad}}\, x^\alpha)^T \in K^{r\times r} \tag{2.6a}$$

$$[n_i, n_k] = 0 \tag{2.6b}$$

$$[x_\alpha, x_\beta] = R^j_{\alpha\beta} n_j \qquad R^j_{\alpha\beta} \in K\,. \tag{2.6c}$$

The matrices A^α are linearly nilindependent and commute pairwise

$$[A^\alpha, A^\beta] = 0 \qquad 1 \leqslant \alpha, \beta \leqslant f\,. \tag{2.7}$$

For $f \geqslant 3$ the matrices A^α and the constants $R^j_{\alpha\beta}$ satisfy

$$R^j_{\alpha\beta} A^\gamma_{jk} + R^j_{\gamma\alpha} A^\beta_{jk} + R^j_{\beta\gamma} A^\alpha_{jk} = 0 \qquad 1 \leqslant k \leqslant r \qquad 1 \leqslant \alpha\beta\gamma \leqslant f\,. \tag{2.8}$$

A classification of the Lie algebras L thus amounts to a classification of the matrices A^α and constants $R^j_{\alpha\beta}$ under the following transformations:

(1) Redefinition of the space F:

$$\tilde{x}_\alpha = x_\alpha + r_{\alpha j} n_j \qquad r_{\alpha j} \in K\,. \tag{2.9a}$$

(2) Change of basis in F:

$$\tilde{\tilde{x}} = G\tilde{x} \qquad G \in GL(f, K)\,. \tag{2.9b}$$

(3) Change of basis in $NR(L)$:

$$\tilde{n} = Sn \qquad S \in GL(r, K)\,. \tag{2.9c}$$

Proof. Equation (2.6) simply expresses known results about the dimension of the nilradical of a solvable Lie algebra, the fact that $NR(L)$ is an ideal, that it is by assumption Abelian and that the derived algebra is contained in the nilradical. Equation (2.7) is a consequence of the Jacobi identities for the triplets $\{x_\alpha, x_\beta, n_j\}$, $1 \leqslant \alpha < \beta \leqslant f$, $1 \leqslant j \leqslant r$. Similarly, (2.8) is a consequence of the Jacobi identities for $\{x_\alpha, x_\beta, x_\gamma\}$, $1 \leqslant \alpha < \beta < \gamma \leqslant f$. The transformations (2.9) leave relations (2.6), (2.7) and (2.8) invariant, but transform the matrices A^α and constants $R^j_{\alpha\beta}$.

More specifically, if we put

$$n'_i = S_{ik} n_k \qquad x'_\alpha = G_{\alpha\beta} x_\beta + r_{\alpha a} n_a \tag{2.10}$$

we obtain the commutation relations (2.6) with A^α and $R^j_{\alpha\beta}$ replaced by

$$A'^\alpha = G_{\alpha\beta}(S A^\beta S^{-1}) \tag{2.11a}$$

$$R'^j_{\alpha\beta} = (G_{\alpha\mu} G_{\beta\nu} R^a_{\mu\nu} + r_{\alpha b} G_{\beta\nu} A^\nu_{ba} - r_{\beta b} G_{\alpha\mu} A^\mu_{ba})(S^{-1})_{aj}\,. \tag{2.11b}$$

This completes the proof.

Theorem 1 provides us with a *general classification procedure* for obtaining all solvable Lie algebras L of the considered type for given values of $n = \dim L$ and $r = \dim NR(L)$. The procedure is:

(1) Classify all Abelian sub-algebras $A_f(r)$ of $gl(r, K)$ of dimension f, containing no nilpotent elements, into conjugacy classes under the action of $GL(r, K)$. Choose a representative of each conjugacy class, i.e. use the transformations (2.11) to transform the matrices A^α, $\alpha = 1, \ldots, f$ into some chosen canonical form.

For $f = 1$ there is just one matrix A and it can be transformed to its Jordan canonical form. For $f = r$ the matrices $\{A^1, \ldots, A^r\}$ form a basis of a Cartan sub-algebra of $gl(r, K)$. As shown below (theorem 3) the algebra L will, in general, then be decomposable ($n > 2$, $K = \mathbb{C}$ and for $n > 4$, $K = \mathbb{R}$).

For $1 < f < r$ the problem of classifying the Abelian sub-algebras $A_f(r) \subset gl(r, K)$ is more difficult. Use should be made of known results on maximal Abelian sub-algebras of $gl(r, K)$ [13–18] (see below).

(2) Determine the structure constants $R^j_{\alpha\beta}$ of (2.6c). For $f = 1$ the question does not arise. For $f \geqslant 2$ we have a mapping from the factor algebra $F = \{x_1, \ldots, x_f\}$ into K^r, i.e. $\vec{R}_{\alpha,\beta} = \{R^j_{\alpha,\beta}\}$ can be viewed as a two-cocycle. Coboundaries, i.e. trivial cocycles, are generated by the transformation (2.9a). They have the form

$$D^j_{\alpha\beta} = r_{\alpha i}\, A^\beta_{ij} - r_{\beta i}\, A^\alpha_{ij}\,. \tag{2.12}$$

Using (2.7), it is easy to check that for $f \geqslant 3$ the coboundaries $D^j_{\alpha\beta}$ satisfy (2.8). The cohomology is trivial if all cocycles are coboundaries. The factor algebra F is then itself an Abelian Lie algebra, i.e. we can set $R^j_{\alpha\beta} = 0$ for all α, β and j.

(3) Weed out the decomposable Lie algebras among the constructed solvable Lie algebras.

We see from theorem 1 and the above discussion that we cannot expect to have a nice closed-form result for solvable Lie algebras of arbitrary dimensions. We shall present further partial results that together make the classification of solvable Lie algebras easy, once the dimensions r and f are fixed. We shall concentrate on the case $K = \mathbb{C}$ but will also point out the differences that occur for the field $K = \mathbb{R}$.

The f matrices A^{α} form an Abelian sub-algebra $A_f(r) \subset gl(r, K)$, containing no nilpotent matrices. Each set $A_f(r)$ is contained in at least one maximal Abelian sub-algebra (MASA) of $gl(r, K)$. A sizable literature exists on MASAs of the classical Lie algebras. For a view of classical results, including the Kravchuk normal form of MASAs of $sl(r, \mathbb{C})$, we refer to Suprunenko and Tyshkevich [13]. More recent results on MASAs of other classical Lie algebras can be found in [14–18].

The pertinent results for our purpose are:

(1) A MASA of $gl(r, K)$ can always be written as

$$\mathrm{MASA}(gl(r, K)) = K I_r \oplus \mathrm{MASA}(sl(r, K)).$$

(2) A MASA of $sl(r, K)$ can be either indecomposable, or decomposable into a direct sum of indecomposable ones.

(3) An indecomposable MASA of $sl(r, \mathbb{C})$ is always a maximal Abelian nilpotent subalgebra (MANS). A MANS is represented by nilpotent matrices in any finite-dimensional representation. A MANS is characterized by a Kravchuk signature

$$(\lambda, \mu, \nu) \qquad 1 \leqslant \lambda \qquad 1 \leqslant \nu \qquad 0 \leqslant \mu \qquad \lambda + \mu + \nu = r \qquad \lambda, \mu, \nu \in \mathbb{Z}$$

and can be transformed to Kravchuk normal form. Thus, all matrices X of a MANS of $sl(r, \mathbb{C})$ can be simultaneously written in the form

$$X = \begin{pmatrix} 0_\lambda & 0 & 0 \\ A & S_\mu & 0 \\ Y & B & 0_\nu \end{pmatrix} \qquad S_\nu = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \tag{2.13}$$

where 0_λ, 0_ν and S_μ are square matrices of the indicated dimension. Commutativity imposes further conditions [13–17] on the matrices A, B and S_μ.

(4) An indecomposable MASA of $sl(r, \mathbb{R})$ can either be absolutely indecomposable (AID) or indecomposable, but not absolutely indecomposable (ID & NAID). The absolutely indecomposable ones are MANS and can be written as in (2.13), but with real entries. Thus they remain indecomposable after field extension. The non-absolutely indecomposable MASA's of $sl(r, \mathbb{R})$, on the other hand, become decomposable after complexification. They exist only for r even and have the form [15]

$$\mathbb{R} \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix} \oplus \mathrm{MASA}(sl(r_0, \mathbb{C})) \qquad r_0 = \frac{r}{2} \tag{2.14}$$

where $sl(r_0, \mathbb{C})$ is represented by the matrices

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1r_0} \\ \vdots & & \vdots \\ \alpha_{r_0 1} & \cdots & \alpha_{r_0 r_0} \end{pmatrix} \qquad \alpha_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix} \qquad a_{ij}, b_{ij} \in \mathbb{R} \qquad \sum_{i=1}^{r_0} a_{ii} = 0. \tag{2.15}$$

The corresponding MANS of $sl(r_0, \mathbb{C})$ can again be written in Kravchuk normal form.

Let us now return to the commuting matrices A^α of (2.6) and (2.11). They form an Abelian sub-algebra of $gl(r, K)$ and hence a sub-algebra of some MASA. This MASA cannot be a MANS; as a matter of fact it contains no nilpotent elements at all. Hence all matrices A_α can be simultaneously brought to the same block diagonal form. Moreover each block can be brought to a triangular form.

We thus arrive at the following theorem.

Theorem 2. Let L be a finite-dimensional solvable Lie algebra over $\mathbb{C}$ with an Abelian nilradical NR. The matrices A^α in (2.6) and in theorem 1 can be simultaneously transformed into a block diagonal form

$$A^\alpha = \begin{pmatrix} T_1^\alpha(a_1^\alpha) & & & & & \\ & \ddots & & & & \\ & & T_p^\alpha(a_p^\alpha) & & & \\ & & & T_{p+1}^\alpha(0) & & \\ & & & & \ddots & \\ & & & & & T_{p+q}^\alpha(0) \end{pmatrix} \tag{2.16a}$$

$$T_j^\alpha(a_j^\alpha) = \begin{pmatrix} a_j^\alpha & & 0 \\ & \ddots & \\ * & & a_j^\alpha \end{pmatrix} \in \mathbb{C}^{r_j \times r_j} \qquad \begin{cases} a_j^\alpha = \delta_j^\alpha & 1 \leqslant j \leqslant f \\ a_j^\alpha \in \mathbb{C} & f+1 \leqslant j \leqslant p \\ a_j^\alpha = 0 & p+1 \leqslant j \leqslant p+q \end{cases} \tag{2.16b}$$

$$\begin{gathered} \sum_{j=1}^{p+q} r_j = r \qquad 1 \leqslant f \leqslant p \leqslant r \qquad 1 \leqslant \alpha \leqslant f \qquad p+q \leqslant r \\ 1 \leqslant r_1 \leqslant r_2 \leqslant \cdots \leqslant r_p \qquad 0 \leqslant r_{p+q} \leqslant r_{p+q-1} \leqslant \cdots \leqslant r_{p+1} . \end{gathered} \tag{2.16c}$$

The off-diagonal elements of each triangular block $T_1^\alpha(a_1^\alpha)$ form an Abelian nilpotent sub-algebra of $sl(r_j, \mathbb{C})$.

In order to obtain all complex solvable Lie algebras of the considered type, we must consider all partitions of r satisfying (2.16c). For each partition we must construct all inequivalent Abelian indecomposable sub-algebras of $sl(r_j, \mathbb{C})$, $1 \leqslant j \leqslant p+q$. □

Comments.

(1) Notice that the Abelian sub-algebras of $sl(r_j, \mathbb{C})$ need not be maximal. They must however be indecomposable, otherwise they would also appear in some other decomposition of r.

(2) The situation for $K = \mathbb{R}$ is conceptually quite similar, however the usual complications arise. Thus, in addition to the non-nilpotent and nilpotent blocks $T_j^\alpha(1)$ and $T_k^\alpha(0)$, respectively, a further type of non-nilpotent block can occur, namely

$$\tilde{T}_j^\alpha(b_j^\alpha, c_j^\alpha) = \begin{pmatrix} b_j^\alpha & c_j^\alpha & & & \\ -c_j^\alpha & b_j^\alpha & & 0 & \\ & & \ddots & & \\ & & & b_j^\alpha & c_j^\alpha \\ & * & & -c_j^\alpha & b_j^\alpha \end{pmatrix} \tag{2.17}$$

with $c_j^\alpha \neq 0$ for at least one value of α. Each block $\tilde{T}_j^\alpha(b_j^\alpha, c_j^\alpha)$ provides either one, or two non-nilpotent matrices A^α. Consider for instance, $f = 2$, $r = 4$. We can have

$$A^1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ * & * & 0 & 1 \\ * & * & -1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}$$

(one block), or

$$A^1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & b_1 & 0 \\ & & 0 & b_1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

(two blocks).

The matrices A^α will have the form

$$A^\alpha = \begin{pmatrix} T_1^\alpha(a_1^\alpha) & & & & & & & & \\ & \ddots & & & & & & & \\ & & T_p^\alpha(a_p^\alpha) & & & & & & \\ & & & \tilde{T}_{p+1}^\alpha(b_{p+1}^\alpha, c_{p+1}^\alpha) & & & & & \\ & & & & \ddots & & & & \\ & & & & & \tilde{T}_{p+q}^\alpha(b_{p+q_1}^\alpha, c_{p+q}^\alpha) & & & \\ & & & & & & T_{p+q+1}^\alpha(0) & & \\ & & & & & & & \ddots & \\ & & & & & & & & T_{p+q+s}^\alpha(0) \end{pmatrix} \tag{2.18a}$$

with $T_j^\alpha(a_j^\alpha)$ as in (2.16*b*), $\tilde{T}_{p+i}^\alpha(b_{p+i}^\alpha, c_{p+i}^\alpha)$ as in (2.17). The linear nilindependence of $A^1, \ldots, A^f$ is assured by setting

$$a_j^\alpha = \delta_j^\alpha \qquad 1 \leqslant j \leqslant p \qquad p + 2q \geqslant f \tag{2.18b}$$

and by appropriately specifying the entries $(b_{p+j}^\alpha, c_{p+j}^\alpha)$, $1 \leqslant j \leqslant q$.

Theorem 2 goes quite far towards a classification of the solvable algebras L with Abelian nilradical $NR(L)$, as can be seen from the following example.

Example. $K = \mathbb{C}$, $\dim L = n = 8$, $\dim NR(L) = r = 5$, $f = 3$. We have 3 matrices $A^1, A^2, A^3 \in \mathbb{R}^{5\times5}$, hence we need at least 3 blocks to assure linear nilindependence. The allowed partitions, values of p and q and corresponding matrices are

(1) $5 = 3 + 1 + 1 \quad (p = 3, s = 0)$

$$A^1 = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ b_1 & c_1 & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & & & & \\ a_2 & 0 & & & \\ b_2 & c_2 & 0 & & \\ & & & 1 & \\ & & & & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & & & & \\ a_3 & 0 & & & \\ b_3 & c_3 & 0 & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix}. \tag{2.19a}$$

Commutativity requires that the three-dimensional block should represent a MASA of $gl(3, \mathbb{C})$. There are 3 possible Kravchuk signatures, for which we have

$$\begin{aligned} &(2\,0\,1) \quad a_\alpha = 0 \\ &(1\,0\,2) \quad c_\alpha = 0 \qquad \alpha = 1, 2, 3 \\ &(1\,1\,1) \quad a_\alpha = c_\alpha . \end{aligned} \tag{2.19b}$$

We mention that here, and in some cases below, further simplifications are possible. Take for example the Kravchuk signature (1 1 1), i.e. $a_\alpha = c_\alpha$. By a change of basis in the nilradical we can transform (a_1, b_1) into $(1, 0)$, $(0, 1)$, or $(0, 0)$. For $a_1 = 1$, $b_1 = 0$ no further simplifications are possible. For $a_1 = 0$, $b_1 = 1$ we can transform (a_2, b_2) into $(a_2, 0)$, $(0, b_2)$ or $(0, 0)$ for $a_2 \neq 0$; $a_2 = 0$, $b_2 \neq 0$, or $a_2 = b_2 = 0$, respectively. In the last case we can take (a_3, b_3) into $(1, 0)$, $(0, 1)$, or $(0, 0)$, as the case may be. Thus, the number and range of parameters can be greatly restricted at the price of splitting each case into many subcases. We shall skip this type of discussion below.

(2) $5 = 2 + 2 + 1 \quad (p = 3,\ s = 0)$

$$A^1 = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ & & 0 & & \\ & & b_1 & 0 & \\ & & & & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & & & & \\ a_2 & 0 & & & \\ & & 1 & & \\ & & b_2 & 1 & \\ & & & & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & & & & \\ a_3 & 0 & & & \\ & & 0 & 0 & \\ & & b_3 & 0 & \\ & & & & 1 \end{pmatrix} . \tag{2.20}$$

(3) $5 = 2 + 1 + 1 + 1$

$$A^1 = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & b_1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & & & & \\ a_2 & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & b_2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & & & & \\ a_3 & 0 & & & \\ & & 0 & & \\ & & & 1 & \\ & & & & b_3 \end{pmatrix}$$

$$\begin{aligned} &p = 3 \quad s = 1 \text{ for } (b_1, b_2, b_3) = (0, 0, 0) \\ &p = 4 \quad s = 0 \text{ for } (b_1, b_2, b_3) \neq (0, 0, 0) . \end{aligned} \tag{2.21}$$

(4) $5 = 5 \times 1$

$$A^1 = \begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & a_1 & \\ & & & & b_1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & a_2 & \\ & & & & b_2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & a_3 & \\ & & & & b_3 \end{pmatrix}$$

$$\begin{aligned} &p = 3 \quad s = 2 \text{ for } a_i = b_i = 0 \\ &p = 4 \quad s = 1 \text{ for } (a_1, a_2, a_3) \neq (0, 0, 0) \quad b_i = 0 \\ &p = 5 \quad s = 0 \text{ for } (a_1, a_2, a_3) \neq (0, 0, 0) \quad (b_1, b_2, b_3) \neq (0, 0, 0) . \end{aligned} \tag{2.22}$$

All four cases above also occur for $K = \mathbb{R}$, but further ones also exist, for example

(5) $5 = 3 + 2 \quad (p = 1,\ q = 1,\ s = 0)$

$$A^1 = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ b_1 & c_1 & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & & & & \\ a_2 & 0 & & & \\ b_2 & c_2 & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & & & & \\ a_3 & 0 & & & \\ b_3 & c_3 & 0 & & \\ & & & 1 & 0 \\ & & & 0 & 1 \end{pmatrix} \tag{2.23}$$

((2.19b) holds for a_i, b_i, c_i).

(6) $5 = 1+2+2 \quad (p=2,\ q=1,\ s=0)$

$$A^1 = \begin{pmatrix} 1 &&&& \\ & 0 & 0 && \\ & a_1 & 0 && \\ &&& b_1 & \\ &&&& b_1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 &&&& \\ & 1 & 0 && \\ & a_2 & 1 && \\ &&& b_2 & \\ &&&& b_2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 &&&& \\ & 0 & 0 && \\ & a_3 & 0 && \\ &&& b_3 & 1 \\ &&& -1 & b_3 \end{pmatrix}. \tag{2.24}$$

(7) $5 = 1+2+2 \quad (p=1,\ q=1,\ s=1)$

$$A^1 = \begin{pmatrix} 1 &&&& \\ & 0 & 0 && \\ & 0 & 0 && \\ &&& 0 & 0 \\ &&& 1 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 &&&& \\ & 0 & 1 && \\ & -1 & 0 && \\ &&& 0 & 0 \\ &&& a_2 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 &&&& \\ & 1 & 0 && \\ & 0 & 1 && \\ &&& 0 & 0 \\ &&& a_3 & 0 \end{pmatrix}. \tag{2.25}$$

(8) $5 = 1+2+2 \quad (p=1,\ q=2)$

$$A^1 = \begin{pmatrix} 1 &&&& \\ & 0 & 0 && \\ & 0 & 0 && \\ &&& a_1 & b_1 \\ &&& -b_1 & a_1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 &&&& \\ & 0 & 1 && \\ & -1 & 0 && \\ &&& a_2 & b_2 \\ &&& -b_2 & a_2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 &&&& \\ & 1 & 0 && \\ & 0 & 1 && \\ &&& a_3 & b_3 \\ &&& -b_3 & a_3 \end{pmatrix}$$
$$(b_1, b_2, b_3) \neq (0,0,0). \tag{2.26}$$

(9) $5 = 1+2+2 \quad (p=1,\ q=2)$

$$A^1 = \begin{pmatrix} 1 &&&& \\ & 0 & 0 && \\ & 0 & 0 && \\ &&& b_1 & 0 \\ &&& 0 & b_1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 &&&& \\ & 0 & 1 && \\ & -1 & 0 && \\ &&& b_2 & 0 \\ &&& 0 & b_2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 &&&& \\ & 0 & 0 && \\ & 0 & 0 && \\ &&& b_3 & 1 \\ &&& -1 & b_3 \end{pmatrix}. \tag{2.27}$$

(10) $5 = 1+2+1+1$

$$A^1 = \begin{pmatrix} 1 &&&& \\ & 0 & 0 && \\ & 0 & 0 && \\ &&& a_1 & \\ &&& 0 & b_1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 &&&& \\ & 0 & 1 && \\ & -1 & 0 && \\ &&& a_2 & \\ &&&& b_2 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 &&&& \\ & 1 & 0 && \\ & 0 & 1 && \\ &&& a_3 & \\ &&&& b_3 \end{pmatrix}$$

$$p=1, \quad q=1, \quad s=2 \text{ for } a_i = b_i = 0 \tag{2.28}$$
$$p=2, \quad q=1, \quad s=1 \text{ for } (a_1,a_2,a_3) \neq (0,0,0) \quad b_i = 0$$
$$p=3, \quad q=1, \quad s=0 \text{ for } (a_1,a_2,a_3) \neq (0,0,0) \quad (b_1,b_2,b_3) \neq (0,0,0).$$

3. Decomposability properties of the solvable Lie algebras

3.1. General comments

So far, nothing guarantees that the Lie algebras L described in theorems 1 and 2 are indecomposable. Indeed, in general, they may be decomposable into direct sums of lower-dimensional Lie algebras, either solvable with Abelian nilradicals, or Abelian,

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_k. \tag{3.1}$$

This type of decomposition can occur in two manners, described below.

3.2. Central decompositions

The algebra L may have a centre $C(L)$. If the centre is not contained in the derived algebra, then the algebra L is decomposable.

Proposition 3. Let $C(L)$ be the centre of L

$$[C(L), L] = 0 \qquad C(L) \subset L \tag{3.2}$$

and let

$$C(L) = C_1(L) \oplus C_2(L) \qquad C_2(L) \subseteq D(L) \qquad C_1(L) \cap D(L) = \emptyset . \tag{3.3}$$

Then

$$L \sim (L/C_1(L)) \oplus C_1(L) \tag{3.4}$$

where $L/C_1(L)$ is itself a solvable Lie algebra with an Abelian nilradical. □

Proof. The centre $C(L)$ is an Abelian ideal in L. Conditions (3.3) assure that the factor algebra $L/C_1(L)$ is a Lie algebra (none of $D(L)$ is removed).

Proposition 4. Let the algebra L of theorems 1 and 2 be such that the set of matrices A^α contains s_0 one-dimensional zero blocks:

$$T_{p+q+j}(0) = 0 \qquad s - s_0 + 1 \leqslant j \leqslant s \tag{3.5a}$$

and let

$$s_0 > \frac{f(f-1)}{2} . \tag{3.5b}$$

Then the algebra is decomposable.

Proof. The existence of the s_0 zero blocks implies that the centre $C(L)$ contains the corresponding s_0 elements n_j, $f - s_0 + 1 \leqslant j \leqslant f$. These elements n_j do not figure in the derived algebra $D(L)$ in (2.6a). The only way they can be contained in $D(L)$ is on the right-hand side of relations (2.6c). Only $f(f-1)/2$ such relations exist, hence at most that many linearly independent elements of $NR(L)$ figure in (2.6c). If (3.5) holds then the centre $C(L)$ contains at least

$$s_1 = s_0 - \frac{f(f-1)}{2} > 0 \tag{3.6}$$

elements, not contained in $D(L)$. By proposition 3, the algebra is decomposable.

3.3. Non-central decomposition

A non-central decomposition into two (or, successively, more) solvable Lie algebras occurs if the matrices A^α of (2.16a) can be split into two sets satisfying

$$\begin{gathered} A^\alpha = \begin{pmatrix} A_1^\alpha & 0 \\ 0 & 0 \end{pmatrix} \qquad A^\beta = \begin{pmatrix} 0 & 0 \\ 0 & A_2^\beta \end{pmatrix} \qquad 1 \leqslant \alpha \leqslant f_0 \\ f_0 + 1 \leqslant \beta \leqslant f \qquad A_1^\alpha \in K^{r_1 \times r_1} \qquad A_1^\beta \in K^{r_2 \times r_2} \qquad r_1 + r_2 = r . \end{gathered} \tag{3.7}$$

The two sets of elements

$$S_1 = \{x_1, \ldots, x_{f_0}, n_1, \ldots, n_{r_1}\} \qquad S_2 = \{x_{f_0+1}, \ldots, x_f, n_{r_1+1}, \ldots, n_{r_1+r_2}\} \tag{3.8}$$

will form mutually commuting Lie algebras if we also have

$$[x_\alpha, x_\beta] = 0 \qquad 1 \leqslant \alpha \leqslant f_0 \qquad f_0 + 1 \leqslant \beta \leqslant f . \tag{3.9}$$

3.4. Nilradicals of minimal dimension

Let us consider the case when

$$r = \dim NR(L) = \frac{1}{2}\dim L = \frac{n}{2} \qquad \text{i.e. } r = f\,.$$

In this case the matrices A^{α}, $\alpha = 1,\ldots,r$ form an r-dimensional Abelian sub-algebra of $gl(r, K)$ containing no nilpotent elements. This is only possible if the algebra $\{A^1,\ldots,A^r\}$ is actually a Cartan sub-algebra of $gl(r, K)$ (a maximal Abelian self-normalizing sub-algebra, i.e. a MASA containing no nilpotent elements).

Consider first the case of $K = \mathbb{C}$. Over the field of complex numbers all Cartan sub-algebras of a semi-simple Lie algebra are mutually conjugate. In particular, the Cartan sub-algebra of $gl(r, \mathbb{C})$ can be represented by r diagonal matrices and we can put

$$(A^{\alpha})_{ik} = \delta_{\alpha i}\delta_{\alpha k} \qquad [n_i, x_\alpha] = \delta_{i\alpha}n_i\,. \tag{3.10}$$

From (2.8) we obtain $R^j_{\beta\gamma} = 0$ for $j \neq \alpha$ or $j \neq \beta$. From (2.11b) with $G = I$, $S = I$, $r_{\alpha\beta} = -R^{\beta}_{\alpha\beta}$, $r_{\beta\alpha} = R^{\alpha}_{\alpha\beta}$ we see that we can set

$$R^j_{\alpha\beta} = 0 \qquad 1 \leqslant \alpha, \beta, j \leqslant r\,. \tag{3.11}$$

Thus the algebra L is indecomposable for $f = r = 1$ and is decomposable into a direct sum of two-dimensional algebras for $r = f \geqslant 2$.

Now consider the algebra L over the field of real numbers $K = \mathbb{R}$. Cartan sub-algebras of the real simple Lie algebras were classified by Kostant [19] and Sugiura [20]. In particular $gl(r, \mathbb{R})$ has $[r/2]+1$ inequivalent classes of Cartan sub-algebras. Each Cartan sub-algebra can be represented by the matrices

$$C_{r_2} = \begin{pmatrix} a_1 & & & & & & & \\ & \ddots & & & & & & \\ & & a_{r_1} & & & & & \\ & & & b_1 & c_1 & & & \\ & & & -c_1 & b_1 & & & \\ & & & & & \ddots & & \\ & & & & & & b_{r_2} & c_{r_2} \\ & & & & & & -c_{r_2} & b_{r_2} \end{pmatrix} \qquad 0 \leqslant r_2 \leqslant \left[\frac{r}{2}\right] \qquad r_1+2r_2 = r\,. \tag{3.12}$$

From (3.12) we see that we can choose the matrices A^{α} to satisfy

$$\begin{aligned} &(A^{\alpha})_{ik} = \delta_{\alpha i}\delta_{\alpha k} && 1 \leqslant \alpha \leqslant r_1 \\ &(A^{\alpha})_{ik} = b_\alpha(\delta_{\alpha i}\delta_{\alpha k} + \delta_{i\alpha+1}\delta_{k\alpha+1}) && \alpha = r_1+1 \quad r_1+3,\ldots,r_1+2r_2-1 \\ &(A^{\alpha})_{ik} = c_\alpha(\delta_{i\alpha-1}\delta_{k\alpha} - \delta_{i\alpha}\delta_{k\alpha-1}) && \alpha = r_1+2 \quad r_1+4,\ldots,r_1+2r_2\,. \end{aligned} \tag{3.13}$$

From relations (2.8) and (2.11b) we see that we can set $R^j_{\alpha\beta} = 0$ for all values of α, β, j.

Let us sum up the results as a theorem.

Theorem 3. Precisely two indecomposable solvable real Lie algebras L with Abelian nilradical NR satisfying

$$n = \dim L = 2 \dim NR = 2r \tag{3.14}$$

exist, namely

$$r = 1 \qquad [n, x] = n \tag{3.15}$$

$$\begin{aligned} r = 2 \qquad & [n_1, x_1] = n_1 \quad [n_1, x_2] = n_2 \quad [n_2, x_1] = n_2 \quad [n_2, x_2] = -n_1 \\ & [x_1, x_2] = 0 \quad [n_1, n_2] = 0 \,. \end{aligned} \tag{3.16}$$

Precisely one indecomposable complex solvable Lie algebra satisfying (3.14) exists; it is *given by the commutation relations (3.15).*

Every other solvable Lie algebra satisfying (3.14) is decomposable into a direct sum of the Lie algebras (3.15) for $K = \mathbb{C}$, and into a direct sum of the Lie algebras of (3.15) and (3.16) for $K = \mathbb{R}$. □

4. Solvable Lie algebras with centres of maximal dimension

Consider again a solvable Lie algebra L with an Abelian nilradical $NR(L)$. An important characteristic of L is the dimension of its centre $C(L)$. Here we shall establish the maximal possible dimension

$$d_{\mathrm{M}} = \max[\dim C(L)] \tag{4.1}$$

compatible with the requirement that the algebra L be indecomposable.

As mentioned above, indecomposability implies

$$C(L) \subseteq D(L) \subsetneq NR(L) \,. \tag{4.2}$$

We also have $C(L) \neq NR(L)$, otherwise L would be nilpotent.

Let us define a matrix $A \in K^{r \times fr}$ obtained by joining together all the matrices A^α of theorems 1 and 2

$$A = (A^1 A^2 \ldots A^f) \in K^{r \times fr} \,. \tag{4.3}$$

An element n_j will be contained in the centre, $n_j \in C(L)$, if the jth row of A consists entirely of zeros. In order to maximize the dimension $\dim C(L)$ of $C(L)$ we must hence maximize the number of zero rows in A. In view of (2.18a) we see immediately that zero rows can only be contained in the nilpotent blocks $T_{p+q+j}(0)$.

To ensure that the matrices A^α be linearly nilindependent in the most economic way we use the first f rows in A, e.g. by choosing A^α to satisfy

$$(A^\alpha)_{ik} = \delta_{\alpha i}\delta_{\alpha k} \qquad 1 \leqslant i \leqslant f \qquad 1 \leqslant k \leqslant r \,. \tag{4.4}$$

We have $r - f$ further rows at our disposal in A. We distinguish two cases.

(1) $r - f \leqslant f(f-1)/2$.

In this case we can set all the remaining rows in A equal to zero

$$(A^\alpha)_{ik} = 0 \qquad f+1 \leqslant i \leqslant r \qquad 1 \leqslant k \leqslant r \tag{4.6}$$

and put

$$[x_\alpha, x_\beta] = n_p \qquad f+1 \leqslant p \leqslant r \tag{4.7}$$

where (4.7) means that each value of p appears for at least one pair (α, β).

In this case we have

$$d_{\mathrm{M}} = r - f\,. \tag{4.8}$$

Notice that the Jacobi identities (2.8) are satisfied identically and that none of the n_p in (4.7) can be annulled by the transformations (2.10).

(2) $r - f > f(f-1)/2$.

We choose the first f rows in A as in (4.4) and the last $f(f-1)/2$ rows to consist entirely of zeros. The commutation relations (4.7) are imposed with

$$[x_\alpha, x_\beta] = n_l \qquad r - \frac{f(f-1)}{2} + 1 \leqslant l \leqslant r\,. \tag{4.9}$$

The rows $f+1 \leqslant i \leqslant r - (f(f-1))/2$ are still at our disposal and their number is

$$N_0 = r - \frac{f(f+1)}{2}\,. \tag{4.10}$$

Let us put

$$N_0 = (f+1)q_0 + \nu \qquad 0 \leqslant \nu \leqslant f \qquad 0 \leqslant q_0 \qquad q_0, \nu \in \mathbb{Z}\,. \tag{4.11}$$

We can now further maximize the dimension of the centre by choosing the remaining blocks in A^α to be of the type $T_{f+j}(0)$, satisfying

$$T_{f+j}(0) \in K^{(f+1)\times(f+1)} \qquad j = 1, \ldots, q_0 \qquad T_{f+q_0+1}(0) \in K^{\nu\times\nu} \qquad \nu \geqslant 2\,. \tag{4.12}$$

For $\nu = 1$ we must add one further non-zero entry on the diagonal of A^α.

We thus have, for $\nu \geqslant 2$

$$A^\alpha = \begin{pmatrix} E^\alpha & & & & & \\ & T_1^\alpha(0) & & & & \\ & & \ddots & & & \\ & & & T_{q_0}^\alpha(0) & & \\ & & & & T_{q_0+1}^\alpha(0) & \\ & & & & & 0_d \end{pmatrix} \qquad E^\alpha \in \mathbb{C}^{f\times f} \qquad (E^\alpha)_{ik} = \delta_{i\alpha}\delta_{k\alpha} \tag{4.13a}$$

$$T_j^\alpha(0) = \begin{pmatrix} 0 & & & \\ \vdots & \ddots & & \\ 0 & \cdots & 0 & \\ a_{j1}^\alpha & \cdots & a_{aj}^\alpha & 0 \end{pmatrix} \in \mathbb{C}^{(f+1)\times(f+1)} \qquad 1 \leqslant j \leqslant q_0$$

$$T_{q_0+1}^\alpha(0) = \begin{pmatrix} 0 & & & \\ \vdots & \ddots & & \\ 0 & & \ddots & \\ a_{q_0+1,1}^\alpha & \cdots & a_{q_0+1,\nu}^\alpha & 0 \end{pmatrix} \in \mathbb{C}^{\nu\times\nu} \qquad d = \frac{f(f-1)}{2}\,. \tag{4.13b}$$

The dimension of the centre is

$$\dim C(L) = \begin{cases} \dfrac{f(f-1)}{2} + fq_0 + \nu - 1 & \text{for } \nu \geqslant 2 \\ \dfrac{f(f-1)}{2} + fq_0 & \text{for } \nu = 0, 1 . \end{cases} \tag{4.14}$$

We replace q_0 using (4.10) and (4.11) and obtain

$$\dim C(L) = \begin{cases} \dfrac{1}{2(f+1)}[2fr - f(f+1) - 2(f+1-\nu)] & \nu \geqslant 1 \\ \dfrac{1}{2(f+1)}[2fr - f(f+1)] & \nu = 0 . \end{cases} \tag{4.15}$$

The condition that the algebra L be indecomposable, i.e. $C(L) \subset D(L)$, requires that the elements

$$n_j \qquad r - \frac{f(f-1)}{2} + 1 \leqslant j \leqslant r \tag{4.16}$$

all occur in the commutation relations (2.6c) and that the matrices

$$M_j = \begin{pmatrix} a_{j1}^1 & \cdots & a_{jf}^1 \\ \vdots & & \\ a_{j1}^f & \cdots & a_{jf}^f \end{pmatrix} \qquad j = 1, \ldots, q_0 \qquad M_{q_0+1} = \begin{pmatrix} a_{q_0+1,1}^1 & \cdots & a_{q_0+1,\nu}^1 \\ \vdots & & \\ a_{q_0+1,1}^f & \cdots & a_{q_0+1,\nu}^f \end{pmatrix} \tag{4.17a}$$

all have maximal rank:

$$\det M_j \neq 0 \qquad j = 1, \ldots, q_0 \qquad \operatorname{rank} M_{q_0+1} = \nu \leqslant f . \tag{4.17b}$$

It is easy to verify that the Jacobi identities (2.8) are satisfied.

The matrices A^α of (4.13) can be subjected to a simultaneous similarity transformation $A^{\alpha\prime} = GA^\alpha g^{-1}$. We choose G in the form

$$A^\alpha = \begin{pmatrix} I_f & & & & & \\ & G_1 & & & & \\ & & \ddots & & & \\ & & & G_{q_0} & & \\ & & & & G_{q_0+1} & \\ & & & & & I_{f(f-1)/2} \end{pmatrix}$$

$$G_j = \begin{pmatrix} M_j & \\ & 1 \end{pmatrix} \qquad j = 1, \ldots, q_0 \qquad G_{q_0+1} = \begin{pmatrix} \tilde{M}_{q_0+1} & \\ & 1 \end{pmatrix} \tag{4.18}$$

where $\tilde{M}_{q_0+1} \in K^{\nu\times\nu}$ is a square matrix containing ν linearly independent rows of M_{q_0+1}. Thus we transform the rows in the matrix (4.13b) into ones satisfying

$$a_{ji}^\alpha = \delta_{\alpha i} \qquad \alpha = 1, \ldots, f \qquad j = 1, \ldots, q_0 \qquad a_{q_0+1i}^\alpha = \delta_{\alpha i} \qquad \alpha = 1, \ldots, \nu . \tag{4.19}$$

The results of this section can now be summed up as a theorem.

Theorem 4. Let L be a solvable non-nilpotent indecomposable Lie algebra of dimension $n = r + f$ with Abelian nilradical NR of dimension r with $f < r < n$. Let $C(L)$ be the centre of L. Define the non-negative integers q_0 and ν by relations (4.10) and (4.11). The maximal possible dimension d_{M} of the centre $C(L)$ is given by

$$d_{\mathrm{M}} = \begin{cases} r - f & \text{if } r - f \leqslant \frac{f(f-1)}{2} \\ \dfrac{1}{2(f+1)}[2fr - f(f+1)] & \text{if } r - f > \frac{f(f-1)}{2} \quad \nu = 0 \\ \dfrac{1}{2(f+1)}[2fr - f(f+1) + 2(\nu - f - 1)] & \text{if } r - f > \frac{f(f-1)}{2} \quad \nu \geqslant 1\,. \end{cases} \tag{4.20}$$

The algebra L, over the fields $K = \mathbb{C}$, or $K = \mathbb{R}$, can be realized as in theorems 1 and 2, with the matrices A^α realized as in (4.13), satisfying (4.19). For $\nu = 1$ the nilpotent block of dimension $\nu \times \nu$ is replaced by $a^\alpha_{q_0+1}$ on the diagonal of A^α with $(a^1_{q_0+1}, \ldots, a^f_{q_0+1}) \neq (0, 0, \ldots, 0)$. The structure constants $R^j_{\alpha\beta}$ in (2.6b) must be such that the commutation relations $[x_\alpha, x_\beta]$ generate the entire subspace (4.16). □

Comments .

(1) The construction of the matrices A^α, described above, is always possible and guarantees the maximal dimension of $C(L)$. It is, however, not necessarily unique. What is unique is the maximal number of zero rows in the matrix A, i.e. the value d_{M} in (4.20).

(2) Over the field $K = \mathbb{R}$ the linear nilindependence of the matrices A^α can be arranged as in (4.13). Alternatively, in the left top corner of A^α we can replace any pair

$$\left\{ \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \right\} \quad \text{by} \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \tag{4.21}$$

to obtain a different Lie algebra L with the same centre $C(L)$ (these algebras are inequivalent over $\mathbb{R}$, equivalent over $\mathbb{C}$).

Let us consider an example with $n = 15$, $f = 3$, $r = 12$. We then have $N_0 = 6$, $q_0 = 1$, $\nu = 2$. We have

$$A^1 = \begin{pmatrix} 1 &&&&&&&&&&& \\ & 0 &&&&&&&&&& \\ && 0 &&&&&&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 1 & 0 & 0 & 0 &&&&& \\ &&&&&&& 0 & 0 &&& \\ &&&&&&& a^1 & 0 &&& \\ &&&&&&&&& 0 && \\ &&&&&&&&&& 0 & \\ &&&&&&&&&&& 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 &&&&&&&&&&& \\ & 1 &&&&&&&&&& \\ && 0 &&&&&&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 1 & 0 & 0 &&&&& \\ &&&&&&& 0 & 0 &&& \\ &&&&&&& a^2 & 0 &&& \\ &&&&&&&&& 0 && \\ &&&&&&&&&& 0 & \\ &&&&&&&&&&& 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 &&&&&&&&&&& \\ & 0 &&&&&&&&&& \\ && 1 &&&&&&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 0 & 0 & 0 &&&&& \\ &&& 0 & 0 & 1 & 0 &&&&& \\ &&&&&&& 0 & 0 &&& \\ &&&&&&& a^3 & 0 &&& \\ &&&&&&&&& 0 && \\ &&&&&&&&&& 0 & \\ &&&&&&&&&&& 0 \end{pmatrix} \tag{4.22}$$

$$(a^1, a^2, a^3) = (1, 1, a), \text{ or } (1, 0, 0) \text{ (up to permutations)}$$

and condition (4.16) is satisfied by putting

$$[x_1, x_2] = n_{10} \qquad [x_2, x_3] = n_{11} \qquad [x_3, x_1] = n_{12}\,. \tag{4.23}$$

5. Special cases and examples

5.1. The case $f = 1$

We have a single element x and hence a single matrix A in theorems 1 and 2. A classification of algebras L is, in this case, given by classifying the Jordan canonical forms of a non-nilpotent matrix $A \in K^{r\times r}$. Let us construct the Lie algebras with centres of maximal dimension.

We have, using (4.10) and (4.11), $r = 2q_0 + 1 + \nu$.

For $\nu = 0$ we have

$$r = 2q_0 + 1 \qquad d_{\mathrm{M}} = \frac{r-1}{2} = q_0 \tag{5.1}$$

and the matrix A of theorem 4 reduces to

$$A = \begin{pmatrix} 1 &&&&& \\ & 0 & 0 &&& \\ & 1 & 0 &&& \\ &&& \ddots && \\ &&&& 0 & 0 \\ &&&& 1 & 0 \end{pmatrix} \tag{5.2}$$

and the centre is $C(L) = (n_2, n_4, \dots, n_{2q_0})$.

The algebra with $d_{\mathrm{M}} = q_0$ is, in this case, unique, both for $K = \mathbb{C}$ and $K = \mathbb{R}$.

For $\nu = 1$ we have

$$r = 2q_0 + 2 \qquad d_{\mathrm{M}} = \frac{r-2}{2} = q_0\,. \tag{5.3}$$

The matrix A of theorem 4 reduces to

$$A = \begin{pmatrix} S &&&&& \\ & 0 & 0 &&& \\ & 1 & 0 &&& \\ &&& \ddots && \\ &&&& 0 & 0 \\ &&&& 1 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & \\ & a \end{pmatrix} \qquad a \neq 0 \tag{5.4}$$

where we have combined the non-zero diagonal entry a corresponding to $\nu = 1$ into the matrix S.

This realization is, however, not unique. The maximal dimension of $C(L)$, given in (5.3) can be achieved in several other ways. More specifically, the entries in the first 4 rows in A of (5.4) can have any of the following forms:

$$A_{0,1} = \begin{pmatrix} 1 & & & \\ & a & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix} \quad a \neq 0 \qquad A_{0,2} = \begin{pmatrix} 1 & 0 & & \\ 1 & 1 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}$$

$$A_{0,3} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & 1 & 0 & \\ & 0 & 1 & 0 \end{pmatrix} \qquad A_{0,4} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & 1 & 0 & \\ & 1 & 0 & 0 \end{pmatrix} \qquad A_{0,5} = \begin{pmatrix} b & 1 & & \\ -1 & b & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix} \tag{5.5}$$

where $A_{0,5}$ is equivalent to $A_{0,1}$ over $\mathbb{C}$, but distinct over $\mathbb{R}$.

5.2. The case $f = 2$

We have $N_0 = r - 3 = 3q_0 + \nu$, $\nu = 0, 1, 2$. Consider first the case (4.5), which occurs for $r \leqslant 3$. For $r = 2$ we have no centre, for $r = 3$ we have $d_{\mathrm{M}} = r - f = 1$ and

$$A^1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \qquad [x_1, x_2] = n_3 . \tag{5.6}$$

Now let us consider $r > 3$. Three possibilities occur:

(1) $\nu = 0 \quad r = 3q_0 + 3 \quad d_{\mathrm{M}} = \frac{1}{3}(2r - 3)$.

Over $\mathbb{C}$ we have just one realization:

$$A^1 = \begin{pmatrix} 1 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & 0 & & & & & & & \\ & & 0 & 0 & & & & & & \\ & & * & * & 0 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0 & & & \\ & & & & & & 0 & 0 & & \\ & & & & & & * & * & 0 & \\ & & & & & & & & & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 0 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 0 & & & & & & & \\ & & 0 & 0 & & & & & & \\ & & * & * & 0 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0 & & & \\ & & & & & & 0 & 0 & & \\ & & & & & & * & * & 0 & \\ & & & & & & & & & 0 \end{pmatrix}$$

$$[x_1, x_2] = n_r . \tag{5.7}$$

Over $\mathbb{R}$, the entries in the first 2 rows can be replaced by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, respectively.

(2) $\nu = 1 \quad r = 3q_0 + 4 \quad d_{\mathrm{M}} = \frac{1}{3}(2r - 5)$.

We put

$$A^1 = \begin{pmatrix} S_1 & & & & & & & & \\ & 0 & & & & & & & \\ & 0 & 0 & & & & & & \\ & * & * & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & 0 & 0 & & \\ & & & & & * & * & 0 & \\ & & & & & & & & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} S_2 & & & & & & & & \\ & 0 & & & & & & & \\ & 0 & 0 & & & & & & \\ & * & * & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & 0 & 0 & & \\ & & & & & * & * & 0 & \\ & & & & & & & & 0 \end{pmatrix}$$

$$[x_1, x_2] = n_r . \tag{5.8}$$

The construction of theorem 4 corresponds to

$$S_1 = \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && a_1 &&& \\ &&& 0 && \\ &&& 0 & 0 & \\ &&& * & * & 0 \end{pmatrix} \qquad S_a = \begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && a_2 &&& \\ &&& 0 && \\ &&& 0 & 0 & \\ &&& * & * & 0 \end{pmatrix} \qquad (a_1, a_2) \neq (0,0)\,. \tag{5.9}$$

The same dimension d_{M} of the centre can be achieved in several other ways, such as by replacing the entries in the first three rows of S^1, S^2 by

$$\left\{ \begin{pmatrix} 1 && \\ & 0 & \\ & b_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 && \\ & 1 & \\ & b_2 & 1 \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 1 && \\ & 0 & a \\ & -a_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 && \\ & 1 & \\ && 1 \end{pmatrix} \right\}$$
$$a \neq 0 \qquad (b_1, b_2) \neq (0,0) \tag{5.10}$$

or putting

$$S_1 = \begin{pmatrix} 1 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ && a_1 & 0 && \\ &&&& 0 & \\ &&&& b_1 & 0 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 0 &&&&& \\ & 1 &&&& \\ && 0 &&& \\ && a_2 & 0 && \\ &&&& 0 & \\ &&&& b_2 & 0 \end{pmatrix}. \tag{5.11}$$

(3) $\nu = 2 \quad r = 3q_0 + 5 \quad d_{\mathrm{M}} = (2r-4)/3$.

Over $\mathbb{C}$ there is just one realization

$$A^1 = \begin{pmatrix} 1 &&&&&&&&&&& \\ & 0 &&&&&&&&&& \\ && 0 &&&&&&&&& \\ && 0 & 0 &&&&&&&& \\ && * & * & 0 &&&&&&& \\ &&&&& \ddots &&&&&& \\ &&&&&& 0 &&&&& \\ &&&&&& 0 & 0 &&&& \\ &&&&&& * & * & 0 &&& \\ &&&&&&&&& 0 && \\ &&&&&&&&& * & 0 & \\ &&&&&&&&&&& 0 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} 0 &&&&&&&&&&& \\ & 1 &&&&&&&&&& \\ && 0 &&&&&&&&& \\ && 0 & 0 &&&&&&&& \\ && * & * & 0 &&&&&&& \\ &&&&& \ddots &&&&&& \\ &&&&&& 0 &&&&& \\ &&&&&& 0 & 0 &&&& \\ &&&&&& * & * & 0 &&& \\ &&&&&&&&& 0 && \\ &&&&&&&&& * & 0 & \\ &&&&&&&&&&& 0 \end{pmatrix}. \tag{5.12}$$

Over $\mathbb{R}$ the first two rows can be replaced in the usual manner to obtain a further algebra L with the same d_{M}.

A result for $f = 2$ that is worth mentioning is:

Proposition 3. Let L be a solvable Lie algebra of dimension n with an Abelian nilradical, satisfying $n = f + r$, $f = 2$. If L has no centre $C(L)$, then the algebra is a semidirect sum of an Abelian factor algebra $\{x_1, x_2\}$ and an Abelian nilradical, i.e. we have

$$[x_1, x_2] = 0\,. \tag{5.13}$$

Proof. If we have $C(L) = 0$ theorem 2 implies that the matrix $A = (A_1 A_2)$ has no zero rows. The matrix $\tilde{A} = (A_1^T A_2^T)^T$ has no zero columns, hence the image space of $\tilde{A}$ is all of NR. From (2.12) with $G = I$, $S = I$ we see that we can annul all coefficients R_{12}^j and obtain (5.13).

6. Conclusions

The results of this article make it quite easy to obtain representative lists of all isomorphy classes of solvable Lie algebras L with Abelian nilradicals for any chosen dimension n. It is, however, obvious that such lists, for $n > 6$ will be very long. The results for $2 \leqslant n \leqslant 6$ are known [7–10]. They can easily be reconstructed, using theorems 1 and 2 of this article. For $n = 2, 3$ we need only consider $f = 1$, for $n = 4, 5, 6$ only $f = 1$ and $f = 2$.

Detailed proofs of all assertions in this article and further results on solvable Lie algebras with Abelian nilradicals are contained in [21], available from the author upon request.

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